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Computers and Mathematics with Applications 53 (2007) 948–960

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Complexity of large-update interior point algorithm for $P_*(\kappa)$ linear complementarity problems

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Received 7 January 2006; accepted 12 December 2006

Abstract

In this paper we propose a new large-update primal–dual interior point algorithm for $P_*(\kappa)$ linear complementarity problems (LCPs). We extend Bai et al.'s primal–dual interior point algorithm for linear optimization (LO) problems to $P_*(\kappa)$ LCPs with generalized kernel functions. New search directions and proximity functions are proposed based on a simple kernel function which is neither a logarithmic barrier nor a self-regular. We show that if a strictly feasible starting point is available, then the new large-update primal–dual interior point algorithms for solving $P_*(\kappa)$ LCPs have $O((1+2\kappa)n \log \frac{n\mu^0}{\varepsilon})$ polynomial complexity which is similar to the polynomial complexity obtained for LO and give a simple complexity analysis. This proximity function has not been used in the complexity analysis of interior point method (IPM) for $P_*(\kappa)$ LCPs before.

Keywords: Primal–dual interior point method; Kernel function; Complexity; Polynomial algorithm; Large-update; Linear complementarity problem

1. Introduction

In this paper we consider the following linear complementarity problem (LCP):

$$\begin{cases} s = Mx + q, \\ xs = 0, \\ x \geq 0, \quad s \geq 0, \end{cases} \tag{LPC}$$

where $x, s, q \in R^n$, $M \in R^{n\times n}$, and xs denotes the componentwise product (Hadamard product) of vectors x and s.

Throughout the paper we assume that $M \in R^{n\times n}$ is a $P_*(\kappa)$ matrix, i.e. for $\kappa \geq 0$, $x \in R^n$,

$$(1+4\kappa) \sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_-(x)} x_i(Mx)_i \geq 0,$$

where

$$J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\} \quad \text{and} \quad J_-(x) = \{i \in J : x_i(Mx)_i < 0\}.$$

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doi:10.1016/j.camwa.2006.12.004

Note that for $\kappa = 0$, $P_*(0)$ is the class of positive semi-definite matrices. This implies that the class of $P_*(\kappa)$ matrices includes the class PSD of positive semi-definite matrices and also includes the class P of matrices with all the principal minors positive.

Linear complementarity problems have many applications in mathematical programming and equilibrium problems. Indeed, it is known that by exploiting the first-order optimality conditions of the optimization problem, any differentiable convex quadratic program can be formulated into a monotone LCP, i.e. $P_*(0)$ LCP, and vice versa [1]. Variational inequality problems are widely used in the study of equilibriums in economics, transportation planning, and game theory. They also have a close connection to the LCPs. The reader can refer to [2] for the basic theory, algorithms, and applications.

The primal–dual IPM for LO was first introduced by Kojima et al. [3] and extended to wider classes of problems, e.g. [4,5]. They first proved the polynomial complexity of the algorithm for LO problem and since then many other algorithms have been developed based on the primal–dual strategy. Since the main feature of the interior point methods (IPMs) is to follow the central path approximately, the existence of the central path is very important. Kojima et al. [6] proved the existence of the central path and generalized the primal–dual interior point algorithm for $P_*(\kappa)$ LCP. They established the same complexity results as in the LO case. Since then a variant of interior point algorithms for LO is generalized to $P_*(\kappa)$ LCPs [6–8].

Most of the polynomial-time interior point algorithms for LO are based on the use of the logarithmic barrier function. Peng et al. [5] introduced a class of self-regular kernel functions and designed primal–dual IPMs based on this class. They obtained the best complexity result for large-update primal–dual IPMs for LO. Bai et al. [9] proposed new primal–dual IPMs for LO based on a simple kernel function and several other kernel functions which are neither logarithmic barrier nor self-regular and obtained the polynomial complexity. Recently Amini and Peyghami [10] imposed some conditions on the self-regular kernel functions and they gave a polynomial complexity analysis for LO. He et al. [7] introduced a self-adjusting IPM for LCPs based on a logarithmic barrier function and gave some numerical experiments.

In this paper we propose a new large-update primal–dual IPM which extends Bai et al.'s algorithm for LO to $P_*(\kappa)$ LCP and give the polynomial complexity $O((1+2\kappa)n \log \frac{n\mu^0}{\varepsilon})$. Since $P_*(\kappa)$ LCP is a generalization of LO problem, our analysis is different from the one in [9,11]. Since our kernel function is not strongly convex, it is not self-regular and hence the analysis is different from the ones in [5,10]. And since our kernel function is not a logarithmic barrier and we use the proximity function based on this kernel function to find a search direction and measure the proximity between the current iterates and the μ-center in the analysis of the algorithm, our analysis is different from the ones in [6,8].

This paper is organized as follows. In Section 2 we recall the basic concepts and define the kernel function and the Algorithm. In Section 3 we describe the growth properties of the proximity function. In Section 4 we compute the feasible step size and the amount of decrease of the proximity function during an inner iteration. Finally, in Section 5 we compute the total number of iterations for our algorithm.

We use the following notations throughout the paper. R^n_+ denotes the set of n-dimensional nonnegative vectors and R^n_{++}, the set of n-dimensional positive vectors. For $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$, $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$, i.e. the minimal component of x, $\|x\|$ is the 2-norm of x, and X is the diagonal matrix from vector x, i.e., $X = \mathrm{diag}(x)$. $x^{\mathrm{T}}s$ is the scalar product of the vectors x and s. e is the n-dimensional vector of ones and I is the n-dimensional identity matrix. J is the index set, i.e. $J = \{1, 2, \ldots, n\}$.

2. Kernel function and the algorithm

In the following we give some definitions about convexity concepts which are essential in our analysis.

Definition 2.1. A twice differentiable function $f : D(\subset R) \to R$ is strongly convex if and only if there exists $m_o > 0$ such that $f''(x) \geq m_o$ for all $x \in D$.

Definition 2.2. A function $f : D(\subset R) \to R$ is exponentially convex if and only if $f(\sqrt{x_1 x_2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$ for all $x_1, x_2 \in D$.

And now we state some well-known results. For proofs and details see the book of Kojima et al. [6].

Proposition 2.3. *If $M \in R^{n\times n}$ is a $P_*(\kappa)$ matrix, then*

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrix $X, S \in R^{n\times n}$.

We use the following corollary to prove that the modified Newton-system has a unique solution.

Corollary 2.4. *Let $M \in R^{n\times n}$ be a $P_*(\kappa)$ matrix and $x, s \in R^n_{++}$. Then for all $a \in R^n$ the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

To get the motivation of the algorithm, we briefly review the generic primal–dual IPMs. The basic idea of primal–dual IPMs is to relax the complementarity condition which is the second equation in the system (LCP) and we get the following parameterized system:

$$\begin{cases} s = Mx + q, \\ xs = \mu e, \\ x > 0, \quad s > 0, \end{cases} \tag{CPP_μ}$$

where $\mu > 0$. Without loss of generality, we assume that (LCP) is strictly feasible, i.e. there exists (x^0, s^0) such that $s^0 = Mx^0 + q$, $x^0 > 0$, $s^0 > 0$. Indeed, we may not have an available strictly feasible point (x^0, s^0). In order to solve this difficulty, we embed (LCP) to an artificial LCP which has a strictly feasible point [6]. Since M is a $P_*(\kappa)$ matrix and (LCP) is strictly feasible, (CPP_μ) has a unique solution for any $\mu > 0$. We denote the solution of (CPP_μ) as $(x(\mu), s(\mu))$ for given $\mu > 0$. We also call it the μ-center for given μ and the solution set $\{(x(\mu), s(\mu)) \mid \mu > 0\}$ the central path of the (LCP). Primal–dual IPMs follow the central path approximately and approach the solution of the (LCP) as μ goes to zero [6]. Moreover, we may assume that we have a starting point $(x, s) > 0$ which is in a certain neighborhood of some μ-center. We then decrease μ to $\mu_+ := (1-\theta)\mu$ for some $\theta \in (0, 1)$ and we solve the following Newton-system by letting $\mu := \mu_+$,

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = \mu e - xs. \end{cases} \tag{2.1}$$

By Corollary 2.4, we get the unique search direction $(\Delta x, \Delta s)$, which is called the Newton search direction. By taking a step along the search direction where the step size is chosen by some line search rules we get a new iterate. We repeat this procedure until the present iterate is in the neighborhood of $(x(\mu), s(\mu))$. Then μ is reduced again by the factor $(1-\theta)$ and we apply Newton's method targeting the new μ-center, and so on. This process is repeated until μ is small enough. To simplify the analysis, we define the following notations:

$$d = \sqrt{\frac{x}{s}}, \qquad v = \sqrt{\frac{xs}{\mu}}, \qquad d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}. \tag{2.2}$$

Then we can transform the system (2.1) to the following scaled Newton-system:

$$\begin{cases} -\bar{M}d_x + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.3}$$

where $\bar{M} = DMD$ and $D = \mathrm{diag}(d)$.

Note that the right-hand side $v^{-1} - v$ of the second equation in (2.3) is the negative gradient of logarithmic barrier functions

$$\Psi_l(v) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right),$$

i.e.

$$d_x + d_s = -\nabla \Psi_l(v). \tag{2.4}$$

We consider a barrier function $\psi(t) : D \to R_+$, with $R_{++} \subseteq D$ in [9] as follows:

$$\psi(t) = t - 1 + \frac{t^{1-q} - 1}{q - 1}, \quad q > 1.$$

To simplify the analysis we will restrict ourselves to the case where the proximity function $\Psi(v)$ is separable with identical coordinate functions. Thus, letting ψ denote the function on the coordinates, we have

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i).$$

We call the univariate function $\psi(t)$ the *kernel function* of the proximity function $\Psi(v)$. In this paper we replace $\Psi_l(v)$ by $\Psi(v)$. Then we get the new search direction by solving the following modified Newton-system for (CPP$_\mu$) in the original space:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v \nabla \Psi(v). \end{cases} \tag{NS}$$

For $\psi(t)$ we have

$$\psi'(t) = 1 - t^{-q}, \qquad \psi''(t) = qt^{-q-1}, \qquad \psi'''(t) = -q(q+1)t^{-q-2}.$$

Since $\psi''(t) > 0$, $\psi(t)$ is strictly convex but not strongly convex. Note that

$$\psi(1) = \psi'(1) = 0, \qquad \psi''' < 0.$$

And due to $\psi(1) = \psi'(1) = 0$, $\psi(t)$ is determined by the second derivative:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\varsigma) \mathrm{d}\varsigma \mathrm{d}\xi.$$

We also use the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) = \frac{1}{2} \parallel \nabla \Psi(v) \parallel = \frac{1}{2} \parallel \mathrm{d}x + \mathrm{d}s \parallel . \tag{2.5}$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e.$$

Then the proximity function $\Psi(v)$ measures the discrepancy between the vectors v and e. We know that all known kernel functions in our reference except [11] have the second-order growth term. But the kernel function defined in [11] has a linear growth term which is the simplest algebraic function.

In the algorithm we assume that a proximity parameter τ and a barrier update parameter θ are given and $\tau = O(n)$ and $0 < \theta < 1$, fixed. The algorithm works as follows. We assume that we are given a strictly feasible point (x, s) which is in a τ-neighborhood of the given μ-center. Then we decrease μ to $\mu_+ = (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$ and then we solve the modified Newton-system (NS) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) that is in a τ-neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1 - \theta$ and we solve the modified Newton-system targeting the new μ_+-center, and so on. This process is repeated until μ is small enough, say until $n\mu \le \varepsilon$. Throughout the paper, we use the proximity function $\Psi(v)$ to find a search direction and to measure the proximity between the current iterates and the μ-center. Then we get the following algorithm.

Algorithm

Input:
  A threshold parameter $\tau > 0$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter θ, $0 < \theta < 1$;
  starting point (x^0, s^0) and $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$;
begin
  $x := x^0$; $s := s^0$; $\mu := \mu^0$;
  while $n\mu \geq \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu$;
    while $\Psi(v) \geq \tau$ do
    begin
      solve Newton-system (NS) for Δx and Δs;
      determine a step size α;
      $x := x + \alpha\Delta x$;
      $s := s + \alpha\Delta s$;
    end
  end
end

Remark 2.5. One distinguishes IPMs as large-update methods when $\theta = \Theta(1)$ and small-update methods when $\theta = \Theta(\frac{1}{\sqrt{n}})$. The small-update methods have the best known iteration complexity, but in practice large-update methods are more efficient than small-update.

3. The growth behavior of the proximity function

In the following lemma we give a key property which is important in the analysis of the algorithm in Section 4.

Lemma 3.1. *The kernel function $\psi(t)$ is exponentially convex.*

Proof. By the definition of $\psi(t)$ and $q > 1$, for all $t > 0$ we have

$$t\psi''(t) + \psi'(t) = (q-1)t^{-q} + 1 > 0.$$

Then by Lemma 1 in [5], $\psi(t)$ is exponentially convex. □

Now we look at the growth behavior of the proximity function $\Psi(v)$. At the start of outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm the largest value of $\Psi(v)$ occurs just after the updates of μ. In the following lemma we give an estimate for the effect of a μ-update on the value of $\Psi(v)$. The reader can refer to Lemma 2.6 in [11] for the proof.

Lemma 3.2. *Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then we have*

$$\Psi(v_+) \leq \frac{1}{\sqrt{1-\theta}}\Psi(v) + \left(\frac{1}{\sqrt{1-\theta}} - 1\right)\frac{qn}{q-1}.$$

By assumption, $\Psi(v) \leq \tau$ just before the update of μ. By Lemma 3.2,

$$\Psi(v_+) \leq \frac{\tau}{\sqrt{1-\theta}} + \left(\frac{1}{\sqrt{1-\theta}} - 1\right)\frac{qn}{q-1} \leq \frac{\tau}{\sqrt{1-\theta}} + \frac{\theta}{\sqrt{1-\theta}}\frac{qn}{q-1},$$

where the second inequality follows from the fact that $1-\sqrt{1-\theta} \le \theta$ for $0 \le \theta < 1$. Define

$$L = \frac{\tau}{\sqrt{1-\theta}} + \frac{\theta}{\sqrt{1-\theta}}\frac{qn}{q-1}.$$

Then by Lemma 3.2, after each μ-update, we have $\Psi(v_+) \le L$. Since $\tau = O(n)$, $q > 1$, and $0 \le \theta < 1$, $L = O(n)$.

4. The step size and the decrease

In this section we compute the feasible step size α such that the proximity function is decreasing and the bound for the decrease. Since $P_*(\kappa)$ LCPs are the generalization of LO problems, we lose the orthogonality of vectors d_x and d_s. So the analysis is different from the one in the LO case. For fixed μ if we take a step size α, then we get new iterates

$$x_+ = x + \alpha \Delta x, \qquad s_+ = s + \alpha \Delta s.$$

Then by (2.2), we have

$$x_+ = x\left(e + \frac{\alpha \Delta x}{x}\right) = x\left(e + \frac{\alpha d_x}{v}\right) = \left(\frac{x}{v}\right)(v + \alpha d_x),$$
$$s_+ = s\left(e + \frac{\alpha \Delta s}{s}\right) = s\left(e + \frac{\alpha d_s}{v}\right) = \left(\frac{s}{v}\right)(v + \alpha d_s).$$

Thus we obtain

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

By Lemma 3.1,

$$\Psi(v_+) = \Psi(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}) \le \frac{1}{2}(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)).$$

Define $f(\alpha)$ as the difference of the new and old proximity function values, i.e.,

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Then we have $f(\alpha) \le f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)) - \Psi(v).$$

Note that $f_1(\alpha)$ is an upper bound for $f(\alpha)$ and

$$f(0) = f_1(0) = 0.$$

By taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha d_{xi})d_{xi} + \psi'(v_i + \alpha d_{si})d_{si}),$$

where d_{xi} and d_{si} denote ith components of the vectors d_x and d_s, respectively. From (2.4) and the definition of δ,

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^{\mathrm{T}}(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^{\mathrm{T}}\nabla\Psi(v) = -2\delta(v)^2. \tag{4.1}$$

By differentiating $f_1'(\alpha)$ with respect to α, we obtain

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha d_{xi})d_{xi}^2 + \psi''(v_i + \alpha d_{si})d_{si}^2). \tag{4.2}$$

Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (NS), for $\Delta x \in R^n$,

$$(1+4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i + \sum_{i\in J_-}\Delta x_i\Delta s_i \geq 0,$$

where $J_+ = \{i \in J : \Delta x_i\Delta s_i \geq 0\}$, $J_- = J - J_+$. Since $d_x d_s = \frac{v^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$ and $\mu > 0$,

$$(1+4\kappa)\sum_{i\in J_+} d_{xi}d_{si} + \sum_{i\in J_-} d_{xi}d_{si} \geq 0. \tag{4.3}$$

For notational convenience we define the following notations:

$$\delta := \delta(v), \qquad \sigma_+ = \sum_{i\in J_+} d_{xi}d_{si}, \qquad \sigma_- = -\sum_{i\in J_-} d_{xi}d_{si}.$$

To estimate the bound for $\|d_x\|$ and $\|d_s\|$, we need the following technical lemma.

Lemma 4.1. $\sigma_+ \leq \delta^2$ *and* $\sigma_- \leq (1+4\kappa)\delta^2$.

Proof. By the definition of σ_+ and σ_-,

$$\sigma_+ = \sum_{i\in J_+} d_{xi}d_{si} \leq \frac{1}{4}\sum_{i\in J_+}(d_{xi}+d_{si})^2 \leq \frac{1}{4}\sum_{i=1}^{n}(d_{xi}+d_{si})^2 = \delta^2.$$

Since M is a $P_*(\kappa)$ matrix, from (4.3)

$$(1+4\kappa)\sigma_+ - \sigma_- \geq 0.$$

Thus

$$\sigma_- \leq (1+4\kappa)\sigma_+ \leq (1+4\kappa)\delta^2. \quad \square$$

In the following lemma we compute the bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 4.2. $\sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\sqrt{1+2\kappa}\,\delta$, *and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\,\delta$.

Proof. Since $\delta = \frac{1}{2}\parallel d_x + d_s \parallel$ and $\sum_{i\in J} d_{xi}d_{si} = \sigma_+ - \sigma_-$,

$$2\delta = \|d_x + d_s\| = \sqrt{\sum_{i=1}^{n}(d_{xi}+d_{si})^2} = \sqrt{\sum_{i=1}^{n}(dx_i^2 + ds_i^2) + 2(\sigma_+ - \sigma_-)}.$$

From (4.3), $(1+4\kappa)\sigma_+ \geq \sigma_-$. Thus we have

$$2\delta \geq \sqrt{\sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2) + 2\left(\frac{1}{1+4\kappa}\sigma_- - \sigma_-\right)} = \sqrt{\sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2) - \frac{8\kappa}{1+4\kappa}\sigma_-}.$$

If we square both sides, we have

$$4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2).$$

By Lemma 4.1,

$$4(1+2\kappa)\delta^2 \geq 4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2).$$

So we have

$$2\sqrt{1+2\kappa}\,\delta \geq \sqrt{\sum_{i=1}^{n}(d_{x_i}^2 + d_{s_i}^2)} \geq \|d_x\|,$$

and by the same way, we get $2\sqrt{1+2\kappa}\,\delta \geq \|d_s\|$. This completes the proof. $\square$

To estimate the step size and the upper bound for the difference of the new and old proximity measures, we need the following two technical lemmas which are modified versions of Lemmas 3.1 and 3.2 in [11].

Lemma 4.3. $f_1''(\alpha) \le 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta)$.

Proof. Since $\|d_x\| \le 2\sqrt{1+2\kappa}\delta$ and $\|d_s\| \le 2\sqrt{1+2\kappa}\delta$,

$$v_i + \alpha d_{xi} \ge v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta \quad \text{and} \quad v_i + \alpha d_{si} \ge v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta.$$

By (4.2), $\psi'''(t) < 0$, and Lemma 4.2,

$$\begin{aligned} f_1''(\alpha) &= \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha d_{xi})d_{xi}^2 + \psi''(v_i + \alpha d_{si})d_{si}^2) \\ &\le \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta)d_{xi}^2 + \psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta)d_{si}^2) \\ &\le 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta). \quad \square \end{aligned}$$

Lemma 4.4. $f_1'(\alpha) \le 0$ *if* α *is such that*

$$-\psi'(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi'(v_{\min}) \le \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{4.4}$$

Proof. Using (4.1), Lemma 4.3, and the assumption,

$$\begin{aligned} f_1'(\alpha) &= f_1'(0) + \int_0^{\alpha} f_1''(\xi)\mathrm{d}\xi \\ &\le -2\delta^2 + 2(1+2\kappa)\delta^2\int_0^{\alpha}\psi''(v_{\min} - 2\xi\sqrt{1+2\kappa}\delta)\mathrm{d}\xi \\ &= -2\delta^2 - \sqrt{1+2\kappa}\delta\int_0^{\alpha}\psi''(v_{\min} - 2\xi\sqrt{1+2\kappa}\delta)\mathrm{d}(v_{\min} - 2\xi\sqrt{1+2\kappa}\delta) \\ &= -2\delta^2 - \sqrt{1+2\kappa}\delta(\psi'(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) - \psi'(v_{\min})) \\ &\le -2\delta^2 + \sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}} = 0. \quad \square \end{aligned}$$

Let $\rho : [0,\infty) \to (0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0,1]$. Then by the definition of $\psi(t)$, for all $t \ge 0$

$$\rho(t) = \left(\frac{1}{2t+1}\right)^{\frac{1}{q}}. \tag{4.5}$$

In the following lemma which is modified from Lemma 3.3 in [11], we compute the feasible step size α such that the proximity measure is decreasing when we take a new iterate for fixed μ.

Lemma 4.5. *The largest step size* α *that satisfies* (4.4) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{4.6}$$

Proof. We want to compute the step size α such that (4.4) holds with α as large as possible. The derivative of the left-hand side in (4.4) with respect to α is

$$2\delta\sqrt{1+2\kappa}\psi''(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) > 0$$

since $\psi'' > 0$. Hence the left-hand side is monotone increasing in α. So the largest possible value of α satisfying (4.4) occurs when

$$-\psi'(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi'(v_{\min}) = \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{4.7}$$

The derivative of the left-hand side in (4.7) with respect to $v_{\min}$ is

$$-\psi''(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi''(v_{\min}) < 0.$$

Since $\psi''' < 0$, the left-hand side in (4.7) is decreasing in $v_{\min}$. This implies that with δ fixed if $v_{\min}$ gets smaller, then α gets smaller. Thus we consider the case of $v_{\min}$ which is the smallest. By the definition of δ and $\Psi(v)$,

$$\delta = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v_i))^2} \geq \frac{1}{2} \mid \psi'(v_{\min}) \mid \geq -\frac{1}{2}\psi'(v_{\min}).$$

Equality holds if and only if $v_{\min}$ is the only coordinate in v which is different from 1 and $v_{\min} \leq 1$, i.e. $\psi'(v_{\min}) \leq 0$. Hence when $v_{\min}$ satisfies

$$-\frac{1}{2}\psi'(v_{\min}) = \delta, \tag{4.8}$$

the smallest step size α occurs. In this case by (4.8) and the definition of ρ,

$$v_{\min} = \rho(\delta). \tag{4.9}$$

From (4.7) and (4.8),

$$-\frac{1}{2}\psi'(v_{\min} - 2\alpha\delta\sqrt{1+2\kappa}) = \delta\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right). \tag{4.10}$$

Then by (4.10) and the definition of ρ,

$$v_{\min} - 2\alpha\delta\sqrt{1+2\kappa} = \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right).$$

Thus by (4.9), the largest step size α is given as follows;

$$\alpha = \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \quad \square$$

In the following lemma we compute the lower bound for $\bar{\alpha}$ defined in Lemma 4.5.

Lemma 4.6. *Let ρ and $\bar{\alpha}$ be as defined in Lemma 4.5. Then for $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$ we have*

$$\bar{\alpha} \geq \frac{1}{q(1+2\kappa)(2\delta+1)^{\frac{1}{q}}(2\delta a + 1)}.$$

Proof. By (4.6) and (4.5),

$$\begin{aligned}\bar{\alpha} &= \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right) \\ &= \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\left(\frac{1}{2\delta+1}\right)^{\frac{1}{q}} - \left(\frac{1}{2(1+1/\sqrt{1+2\kappa})\delta+1}\right)^{\frac{1}{q}}\right) \\ &= \frac{1}{2\delta\sqrt{1+2\kappa}(2\delta+1)^{\frac{1}{q}}}\left(1 - \left(1 - \frac{2\delta/\sqrt{1+2\kappa}}{2(1+1/\sqrt{1+2\kappa})\delta+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

By Bernoulli inequality,

$$\left(1 - \frac{2\delta/\sqrt{1+2\kappa}}{2(1+1/\sqrt{1+2\kappa})\delta+1}\right)^{\frac{1}{q}} \leq 1 + \frac{1}{q}\left(\frac{-2\delta/\sqrt{1+2\kappa}}{2(1+1/\sqrt{1+2\kappa})\delta+1}\right).$$

Hence

$$\bar{\alpha} \geq \frac{1}{2\delta\sqrt{1+2\kappa}(2\delta+1)^{\frac{1}{q}}}\left(1-\left(1+\frac{1}{q}\left(\frac{-2\delta/\sqrt{1+2\kappa}}{2(1+1/\sqrt{1+2\kappa})\delta+1}\right)\right)\right)$$
$$= \frac{1}{q(1+2\kappa)(2\delta+1)^{\frac{1}{q}}(2\delta a+1)}. \quad \square$$

Define

$$\tilde{\alpha} = \frac{1}{q(1+2\kappa)(2\delta+1)^{\frac{1}{q}}(2\delta a+1)}, \tag{4.11}$$

where $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$. We will use $\tilde{\alpha}$ as the default step size in our algorithm. By Lemma 4.6 we have $\bar{\alpha} \geq \tilde{\alpha}$. We cite the following result for the following lemma and the reader can refer to Lemma 12 in [5] for the proof.

Lemma 4.7. *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{h'(0)t}{2}, \quad 0 \leq t \leq t^*.$$

In the following lemma, the modified version of Lemma 3.6 in [11], we evaluate the decrease of the proximity function value.

Lemma 4.8. *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Proof. Define the univariate function h as follows:

$$h(0) = f_1(0) = 0, \qquad h'(0) = f_1'(0) = -2\delta^2, \qquad h''(\alpha) = 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta).$$

Then since $\psi'' > 0$, $h(\alpha)$ is strictly convex. By Lemma 4.3, $f_1''(\alpha) \leq h''(\alpha)$. So we have, $f_1'(\alpha) \leq h'(\alpha)$ and $f_1(\alpha) \leq h(\alpha)$. Taking $\alpha \leq \bar{\alpha}$, with $\bar{\alpha}$ as defined in Lemma 4.5 and using the fundamental theorem of calculus, we have

$$\begin{aligned} h'(\alpha) &= h'(0) + \int_0^\alpha h''(\xi)\mathrm{d}\xi \\ &= -2\delta^2 + 2(1+2\kappa)\delta^2 \int_0^\alpha \psi''(v_{\min} - 2\xi\sqrt{1+2\kappa}\delta)\mathrm{d}\xi \\ &= -2\delta^2 - \sqrt{1+2\kappa}\delta(\psi'(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta) - \psi'(v_{\min})) \\ &\leq -2\delta^2 + \sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}} = 0. \end{aligned}$$

Then $h(\alpha)$ attains its minimum at $\alpha = \bar{\alpha}$. Since $h'''(\alpha) = -4(1+2\kappa)^{\frac{3}{2}}\delta^3\psi'''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta)$ and $\psi''' < 0$, $h''(\alpha)$ is increasing in α. By Lemma 4.7, for $\alpha \in [0, \bar{\alpha}]$ we have

$$f_1(\alpha) \leq h(\alpha) \leq \frac{1}{2}h'(0)\alpha = -\alpha\delta^2.$$

Since $f(\alpha) \leq f_1(\alpha)$, the proof is completed. $\square$

In the following theorem, we get the upper bound for the difference $f(\alpha)$ between the new and old proximity measures during an inner iteration by Lemma 4.8 and (4.11).

Theorem 4.9. *Let $\tilde{\alpha}$ be a step size as defined in (4.11). Then for $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$ we have*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{q(1+2\kappa)(2\delta+1)^{\frac{1}{q}}(2\delta a+1)}. \tag{4.12}$$

Define $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. The following theorem provides a lower bound on $\delta(v)$ in terms of the proximity function $\Psi(v)$. We state the theorem without proof since the proof is similar to Theorem 4.9 in [9].

Theorem 4.10. *Let $\delta(v)$ be the norm-based proximity measure as defined in (2.5). Then we have*

$$\delta(v) \geq \frac{1}{2}\psi'(\varrho(\Psi(v))).$$

By Theorem 4.10 and the definition of ψ with $\Psi := \Psi(v)$, $\delta := \delta(v)$,

$$\delta \geq \frac{1}{2}\psi'(\varrho(\Psi)) = \frac{1}{2}\left(1 - \frac{1}{\varrho(\Psi)^q}\right). \tag{4.13}$$

Let $s = \psi(t)$, $t \geq 1$. Then by the definition of ϱ, $\varrho(s) = t$.

$$1 + s = 1 + \psi(t) = t + \frac{t^{1-q} - 1}{q - 1}. \tag{4.14}$$

Since $t \geq 1$ and $q > 1$, $t^{q-1} \geq 1$. Hence

$$\frac{t^{1-q}}{q-1} \leq \frac{1}{q-1}. \tag{4.15}$$

By (4.14) and (4.15), $1 + s \leq t = \varrho(s)$. So $1 + \Psi \leq \varrho(\Psi)$ and hence $\frac{1}{\varrho(\Psi)^q} \leq \frac{1}{(1+\Psi)^q}$. Assuming $\Psi \geq \tau \geq 1$, we have

$$\frac{1}{\varrho(\Psi)^q} \leq \frac{1}{(1+\Psi)^q} \leq \frac{1}{2^q}. \tag{4.16}$$

By (4.13) and (4.16), and $q > 1$,

$$\delta(v) > \frac{1}{2}\left(1 - \frac{1}{2^q}\right) > \frac{1}{4}. \tag{4.17}$$

5. Complexity analysis

In this section we analyze the complexity of the algorithm. We cite the following lemma in [5].

Lemma 5.1. *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lfloor \frac{t_0^\gamma}{\beta\gamma} \rfloor$.

Lemma 5.2. *The function*

$$g(\delta) = -\frac{\delta^2}{(2\delta+1)^{\frac{1}{q}}(2\delta a + 1)}$$

is monotone decreasing in δ.

Proof. We will show that $-g(\delta)$ is monotone increasing in δ. By taking derivative $-g(\delta)$ with respect to δ, we have

$$\frac{2\delta(2\delta+1)^{\frac{1}{q}}(2\delta a+1) - \delta^2\left\{\frac{2}{q}(2\delta+1)^{\frac{1}{q}-1}(2\delta a+1) + 2a(2\delta+1)^{\frac{1}{q}}\right\}}{(2\delta+1)^{\frac{2}{q}}(2\delta a+1)^2}. \tag{5.1}$$

Since the denominator in (5.1) is positive, we only check the sign of numerator. If we simplify the numerator, then we get the following:

$$2\delta(2\delta+1)^{\frac{1}{q}}\frac{2a\delta^2(q-1)+(aq+2q-1)\delta+q}{q(2\delta+1)}. \tag{5.2}$$

Since $q>1$, (5.2) is positive and hence $g(\delta)$ is monotone decreasing in δ. □

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as $\Psi_k, k=1,2,\ldots$. Let K denote the total number of inner iterations in the outer iteration. Then we have

$$\Psi_0=O(n),\qquad \Psi_{K-1}>\tau,\quad 0\le\Psi_K\le\tau. \tag{5.3}$$

In the following lemma, we compute the upper bound for the total number of inner iterations which we needed to return to the τ-neighborhood, i.e. $\Psi(v)\le\tau$ after a μ-update.

Lemma 5.3. *Let K be the total number of inner iterations in the outer iteration which is defined in (5.3). Then for $a=1+\frac{1}{\sqrt{1+2\kappa}}$ we have*

$$K\le\left(\frac{3}{2}\right)^{\frac{1}{q}}8q(1+2\kappa)(a+2)\left(\frac{\tau}{\sqrt{1-\theta}}+\frac{\theta}{\sqrt{1-\theta}}\frac{qn}{q-1}\right). \tag{5.4}$$

Proof. By (4.12) and (4.17), and Lemma 5.2,

$$f(\tilde{\alpha})\le-\left(\frac{2}{3}\right)^{\frac{1}{q}}\frac{1}{8q(1+2\kappa)(a+2)},$$

where $a=1+\frac{1}{\sqrt{1+2\kappa}}$. Thus it follows that

$$\Psi_{k+1}\le\Psi_k-\beta{\Psi_k}^{1-\gamma},\quad k=0,1,2,\ldots,K-1,$$

where $\beta=(\frac{2}{3})^{\frac{1}{q}}\frac{1}{8q(1+2\kappa)(a+2)}$ and $\gamma=1$. Hence by Lemma 5.1, the total number K of inner iterations is bounded above by

$$K\le\left(\frac{3}{2}\right)^{\frac{1}{q}}8q(1+2\kappa)(a+2)\left(\frac{\tau}{\sqrt{1-\theta}}+\frac{\theta}{\sqrt{1-\theta}}\frac{qn}{q-1}\right).$$

This completes the proof. □

The upper bound for the total number of iterations is obtained by multiplying the number K by the number of central path parameter updates. If the central path parameter μ has the initial value μ^0 and is updated by multiplying $1-\theta$, with $0<\theta<1$, then after at most

$$\left\lceil\frac{1}{\theta}\log\frac{n\mu^0}{\varepsilon}\right\rceil \tag{5.5}$$

iterations we have $n\mu\le\epsilon$ [12]. In the following theorem we get an upper bound for the total number of iterations.

Theorem 5.4. *Let a linear complementarity problem for any $P_*(\kappa)$ matrix M be given, where $\kappa\ge0$. Assume that a strictly feasible starting point is available with $\Psi(x^0,s^0,\mu^0)\le\tau$ for some $\mu^0>0$. Then the total number of iterations to have a feasible solution such that $n\mu\le\varepsilon$ is bounded by*

$$\left\lceil\left(\frac{3}{2}\right)^{\frac{1}{q}}8q(1+2\kappa)(a+2)\left(\frac{\tau}{\sqrt{1-\theta}}+\frac{\theta}{\sqrt{1-\theta}}\frac{qn}{q-1}\right)\right\rceil\left\lceil\frac{1}{\theta}\log\frac{n\mu^0}{\varepsilon}\right\rceil.$$

Remark 5.5. For large-update methods with $\tau=O(n)$ and $\theta=\Theta(1)$ we get the polynomial complexity $O((1+2\kappa)n\log\frac{n\mu^0}{\varepsilon})$ which is the similar complexity for LO.

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